Optimal Portfolio Allocation with Long-Short Strategies: Application to Factor Investing

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Abstract

This paper examines portfolio allocation optimality and the corresponding efficient frontiers when using long-short strategies. Then, in the context of investing, we compare several portfolio frontiers corresponding to different constraints on portfolio weights. For a benchmark case such as the Fama-French frontier, we explicitly provide the optimal portfolio weights taking into account long-short positions, within the mean-variance framework. This allows us to accurately examine the proximity of efficient frontiers with and without the use of long-short strategies, by examining the impact of correlations. To this end, we also provide explicit spreads between two Markowitz's frontiers when one is dominated by the other. As a by-product, we justify why the efficient frontier generated by the market and the five Fama-French long-short factors is very close to that obtained from the market and the ten short and long legs of each of these five factors.

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1 Introduction

Factor investing was introduced by practitioners as a by-product of factor models of asset pricing. The main risk factor corresponds to the market factor, as emphasized by the capital asset pricing model (CAPM). However, since its introduction, practitioners and academic researchers have realized that a single factor model has its limitations and that there are other systematic sources of risk that might explain the cross-section of expected stock returns. Indeed, over the past fifty years, researchers have begun to identify characteristics that can be associated with the cross-section of stock returns beyond market beta. In this respect, Basu (1977) and Banz (1981) are the two best known pioneering works. They respectively introduce the value and the size effects. Based on empirical studies, additional factors have been introduced. In this regard, the contributions of Fama and French (1992, 1993, 1996, 2015, to name a few of their papers) are landmark (see also Carhart, 1997, Asness *et al*, 2013, 2014, for the momentum effect)[1]. Using the framework of the arbitrage pricing theory, Fama and French (1992, 1993, 1996) have analyzed portfolios based on size and value as risk factors. They build two self financing portfolios, *i.e.* two long/short portfolios, SMB and HML[2] which "mimic combinations of two underlying risk factors or state variables that are of special hedging concern to investors". But it should be noted that there is still controversy surrounding this interpretation. For example, Daniel and Titman (1997) reject the risk factor interpretation and argue that it is the characteristic which explains a higher expected return and not the covariance with the factor. Put another way, it is a matter of determining whether the premia associated with a characteristic are due to the characteristic itself or to the fact that the characteristic is an approximation of an unknown or latent risk factor[3]. More recently, Kozak *et al.* (2018) argue that "horse races between “characteristics” and “covariances” cannot discriminate between alternative models of investor beliefs". In this work, we follow the view of Cornell (2020) who argues that, despite the great importance of the distinction between characteristics and covariances for asset pricing theory, it is of lesser importance to investors.

Koedijk *et al.* (2016a) distinguish three main approaches to introducing factors into portfolio allocation. The first simply seeks to better estimate the factor exposures of a given portfolio allocation, and to take this into account when modifying the allocation. The second and main approach, called the "factor tilting", modulates a portfolio’s exposure to certain factors, either by introducing factor biases into the assets or by adding "pure" factors as new assets. Finally, instead of asset-based allocation, we can consider factor-based allocation where asset classes are replaced by factors. The main novelty of factor investing, apart from the benefits of diversification between factors, is that long-short portfolios associated with factors are able to capture risk premiums. This is where factor investing can broaden the asset universe by adding these risk factors to existing market risk factors, which are simply asset classes such as equities and bonds. The aim is then

[1] See also anomalies emphasized by Jegadeesh and Titman (1993) on the momentum effect.

[2] The SMB, Small minus Big, portfolio return is equal to the difference between the return on a portfolio of small stocks and the return on a portfolio of large stocks. The HML, High minus Low, portfolio return is equal to the difference between a portfolio of high-book-to-market stocks and the return of a portfolio of low-book-to-market stocks.

[3] Note that Fama and French (1993, 1996) argue that the SMB and HML factors capture compensation for distress risk which is of special hedging concern to investors.

to generate more returns in the long run. As Ang (2014) explains, "it is precisely because factors episodically lose money in bad times that there is a long-run reward for being exposed to factor risk. Factor premiums are rewards for investors enduring losses during bad times." Since Ang's (2014) findings on active portfolio management, factor investing has been highlighted for long-term investments (see e.g. Cazalet and Roncalli, 2014; Jurczenko *et al.*, 2015; Brière and Szafarz, 2017b).

The poor performance of standard diversification between financial asset classes during recent crises has made factor-based allocation increasingly popular with institutional investors looking to effectively diversify their portfolios (Koedijk *et al.*, 2016b; Ung and Kang, 2015). Indeed, factor investing can deliver a higher expected return than the market for a given level of volatility. The advantages and disadvantages of factor-based allocation are the subject of much debate (Idzorek and Kowara, 2013; Ilmanen and Kizer, 2012; Arnott *et al.*, 2019). One of the key questions is to determine the optimal allocation between these factors. We can, for example, directly examine several specific portfolios, such as the standard equally weighted portfolio or the portfolio with equal-volatility weighted factors (see e.g. Bender *et al.* 2010; Ilmanen and Kizer, 2012). Different decision criteria can also come into play. For example, we can rely on the global minimum-variance portfolio and analyze efficient frontiers based on mean-variance optimizations (see for example Clarke et al., 2005; Idzorek and Kowara, 2013; Brière and Szafarz, 2017b). Dimson *et al.* (2017) estimate the risk premiums resulting from factor investing over very long periods (up to 117 years) and across many markets (up to 23). They highlight the long-term profitability of strategies based on market capitalization, value versus growth, dividend yield, stock-return momentum, and low-volatility investing.

As shown by Israel and Moskowitz (2013), and Asness *et al.* (2013, 2014), long and short factors legs improve the overall financial performance. In this context, Brière and Szafarz (2017a, 2021) examine the profitability of multifactor portfolios in the US stock market. They ask the following key question: do the excess returns associated with holding factor-based portfolios imply higher risks, and if so, are the excess risks eliminated by factor diversification? They examine how the mean-variance performance of factor investing depends on restrictions on short selling. In addition to the market, they consider ten factors that are defined as the long and short legs of the five factors in the Fama-French (2015) model: (1) small, (2) big, (3) value, (4) growth, (5) robust profitability, (6) weak profitability, (7) conservative investment, (8) aggressive investment, (9) high momentum, (10) low momentum. They show that, when short selling is not restricted, factor investing outperforms sector investing in all respects. However, for long-only portfolios, there is a trade-off between the risk premium associated with factors and the diversification potential of sectors. They also show that, for long-only portfolios, factor investing tends to be more profitable during bullish periods, whereas it is less attractive during bearish periods. Dichtl *et al.* (2021) also show that investment portfolios can be efficiently diversified using factor-based allocation strategies. Bessler *et al.* (2021) compare factor investing with sector allocations for different portfolio weightings such as for example equal-weights, optimal weights with respect to the mean-variance criterion or risk parity. Focusing on the period from May 2007 to November 2020, they find that factor investing offers superior relative performance. Like Brière and Szafarz (2017a, 2021), they also point out that in "normal" times, factor portfolios clearly dominate sector portfolios, while in times of crisis, sector portfolios are superior and offer better diversification opportunities.[4]

Factor investing is usually based on asset pairing[5]: for every stock with a positive type of factor attribute, there is another stock with the negative side of that attribute. For example, for every valuable stock that is cheap relative to its intrinsic value, there must be a stock that is relatively expensive. Long-only factor investing is about taking only one side. A long-short value strategy, on the other hand, involves simultaneously holding long positions in the cheapest stocks and short positions in the most expensive stocks. A long-short fund seeks to eliminate market exposure, or at least to become market neutral. Long-only value strategies do not, by construction, use leverage. Long-short strategies use leverage to reduce correlation with market returns. Leverage may have explicit limits, for example in a $130-30$ mutual fund which can hold short positions of up to 30% of the portfolio. Unconstrained long-short strategies seek to eliminate market exposure by targeting a particular level of risk. For example, a long-short strategy may short as many growth stocks and as many value stocks as necessary to achieve a target volatility level. The main novelty of factor investing, apart from the benefits of diversification between factors, is that long-short portfolios associated with factors are able to capture risk premiums. Indeed, factor investing can allow to provide a higher expected return than the market for a given level of volatility. As shown by Israel and Moskowitz (2013), and Asness *et al.* (2013, 2014), both the long and short legs of factors enhance the overall financial performance.

Brière and Szafarz (2017a,b, 2021) also compare the efficient frontier obtained by constructing portfolios invested 100% in the market index and in optimized proportions of the five long-short factors of Fama-French (2015), the FF benchmark case, with the efficient frontier allowing fully unconstrained optimization. This benchmark portfolio strategy can be analyzed as an extreme[6] case of a core-satellite investment strategy. On one hand, the market index is clearly identified as the passive part of the portfolio, *i.e.* the core. On the other hand, satellites are generally designed to deliver higher risk-adjusted returns than the core and should not be correlated with it. The five Fama-French factors are well suited to this role. Indeed, the risk premium associated with each factor generate excess returns relative to the market and their correlations with the market are negative 4 times out of 5, and very low in the last case (the Small minus Big (SMB) factor, as expected). Surprisingly, Brière and Szafarz (2017b) find that "the performances of the factor portfolios that Fama and French built for asset pricing purposes are remarkably similar to those of optimized long-short portfolios, except for low levels of volatility." This result is shown empirically.

In this paper, we first study from a theoretical point of view the special case corresponding to 100% investment in the market and any weights on long-short factors. Within the mean-variance framework, we provide explicit optimal weights for any expected return objective, allowing us, for example, to explicitly determine the so-called Fama-French efficient frontier with 100% on the market. Using previous results, we can precisely study the proximity of the efficient frontiers with and without the use of long-short strategies, and highlight the key impact of correlations,

[4] See also Adrian et al (2015) who show that the SMB risk premium can be statistically close to zero on average, but varies significantly over time.

[5] See for instance Lin et al (2021).

[6] We use the term "extreme" to mean that we are 100% invested in the market index and globally 0% in the five Fama-French risk factors.

especially when the latter are close to 0.9. Firstly, using a numerical approach, we show that the minimum gap between these two Markowitz's frontiers is most often reached around 0.9. To better appreciate this result, we also provide the explicit gaps between two Markowitz's frontiers when one is dominated by the other.[7] We thus identify the origin of the proximity of the two curves, observed empirically by Brière and Szafarz (2017b). In a second step, we conduct an empirical analysis to compare different portfolio frontiers under different portfolio constraints. To do this, as in Brière and Szafarz (2017a,b, 2021), we recover the previous ten factors from the data retrieved from Kenneth French's website. Indeed, to relax the need for short-selling in factor investing, we proceed in two steps. First, we disentangle the long and short legs of the five historical factors. The resulting ten long-only factors offer additional flexibility in portfolio management. Secondly, any short-selling restrictions are imposed separately on each of these ten factors. We use monthly gross total return (in USD) for all variables over the period July 1963-December 2019. Instead of investing in a stock market index and in a safer asset (T-bills or Bonds), we consider here portfolios invested in the riskless asset (T-bills) and in the tangency portfolio (*i.e.,* the maximum Sharpe ratio) obtained from an efficient frontier built from an investment universe composed of the market and of the ten factors mentioned above. We estimate their correlations during recession periods and non-recession periods. We find that they are most often close to 0.9 since the average correlation between the ten legs during a recession is 0.942 and 0.911 during a normal period (non-recession period). Over the whole of history, the average correlation between the ten legs is equal to 0.917. As expected, the correlation increases slightly during recessions, but the effect is not very significant. In line with the theoretical and numerical analysis, the level of correlations is sufficient to justify the proximity of the efficient frontiers. We also assess the robustness of this proximity by examining the minimum deviation when considering the market and two pairs of factors.

The paper proceeds as follows. Section 2 provides the explicit result about the determination of the efficient frontier when combining the market index (weighted by 1) with optimized proportions of other factors. We also examine and illustrate the closeness of the Markowitz's optimal frontiers, using among others the explicit value of the spread between them. Section 3 is devoted to the empirical illustration of factor investing on long and short legs. It provides various portfolio frontiers corresponding to different constraints on portfolio weights. Finally, Section 4 concludes.

[7] For example, the other Markowitz's frontier corresponds to a case where we invest in a smaller number of financial assets than in the first case.

2 Mean-variance analysis with specific long-short strategies

In what follows, we examine the case of specific long-short strategies drawing parallels with standard mean-variance analysis. The first factor corresponds to the market portfolio itself, while the other factors include, for example, the five factors of Fama-French (2015) (see Brière and Szafarz, 2017a,b, 2021). Within the mean-variance framework, we can first consider the case where there is no constraint on the weights such as no short selling condition. In this case, we can use the Markowitz's results directly (see Markowitz, 1952, 1959) but apply them to factors rather than to standard assets (see an illustration of these solutions in the empirical section, which corresponds to Fama-French long-short factors, namely the FF Benchmark).

The following theoretical results allow us, for example, to explicitly determine the so-called Fama-French efficient frontier with 100% on the market. In this framework, to provide the explicit solutions and illustrate their properties, we consider the case where the investor's weight w_1 on the first factor is fixed and equal to 1. The other components correspond to long-short positions with null initial values. Since we want to consider the market on the one hand and several pairs of assets on the other, we assume that the number of risky assets is odd: $n = 2m + 1$. Denote the rate of return on asset i by R_i . For $k \in \{2, ..., m + 1\}$, denote by $\widetilde{w}_k$ the weight invested on $R_{2k-1} - R_{2k-2}$. The portfolio return is given by:

$$R_P = R_1 + \sum_{k=2}^{m+1} \widetilde{w}_k \left(R_{2k-1} - R_{2k-2}\right) \tag{1}$$

Definition 1 *We set $X_1 = R_1$. For any $k \geq 2$, the long-short position X_k is defined as the difference between the two yields $R_{2k-1} - R_{2k-2}$ (for example, Small minus Big (SMB)).*

From 1, we deduce that, contrary to standard Markowitz's model, we have to invest on the market portfolio X_1 and on the following long-short positions:

$$X_2 = R_3 - R_2, ..., X_{m+1} = R_n - R_{n-1}$$

2.1 Case with only risky assets

We begin by examining the case where all assets are risky. Denote

$$\widetilde{\mathbf{w}} = \begin{pmatrix} \widetilde{w}_2 \\ . \\ . \\ . \\ \widetilde{w}_{m+1} \end{pmatrix}, \ \overline{\mathbf{X}} = \begin{pmatrix} \mathbb{E}\left[X_2\right] \\ . \\ . \\ . \\ \mathbb{E}\left[X_{m+1}\right] \end{pmatrix}, \ \widetilde{\mathbf{s}} = \begin{pmatrix} \sigma^X_{12} \\ . \\ . \\ . \\ \sigma^X_{1(m+1)} \end{pmatrix} \text{ and}$$

$$\widetilde{\mathbf{V}} = \begin{pmatrix} \sigma^X_{22} & . & . & . & \sigma^X_{2(m+1)} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ \sigma^X_{(m+1)2} & . & . & . & \sigma^X_{(m+1)(m+1)} \end{pmatrix}$$

The Lagrangian of the optimization problem is given by:

$$\widetilde{L}\left(\widetilde{\mathbf{w}},\lambda\right)=\sigma_{11}+2\widetilde{\mathbf{w}}'.\widetilde{\mathbf{s}}+\widetilde{\mathbf{w}}'\widetilde{\mathbf{V}}\widetilde{\mathbf{w}}+\lambda\left(\mathbb{E}\left[R_P\right]-\mathbb{E}\left[R_1\right]-\widetilde{\mathbf{w}}'\overline{\mathbf{X}}\right) \tag{2}$$

where λ is the Lagrange multiplier.

The necessary conditions of the first order are:

$$\frac{\partial\widetilde{L}\left(\widetilde{\mathbf{w}},\lambda\right)}{\partial\widetilde{\mathbf{w}}} = 2\widetilde{\mathbf{s}}+2\widetilde{\mathbf{V}}\widetilde{\mathbf{w}}-\lambda\overline{\mathbf{X}}=0 \tag{3}$$

$$\frac{\partial\widetilde{L}\left(\widetilde{\mathbf{w}},\lambda\right)}{\partial\lambda} = \mathbb{E}\left[R_P\right]-\mathbb{E}\left[R_1\right]-\widetilde{\mathbf{w}}'\overline{\mathbf{X}}=0 \tag{4}$$

By assumption, the variance-covariance matrix, $\widetilde{\mathbf{V}}$, is assumed to be invertible. Moreover, this matrix is positive definite, which implies that the previous optimality conditions are necessary and sufficient to obtain a global minimum. The solution of equation (3) is given by:

$$\widetilde{\mathbf{w}}=\widetilde{\mathbf{V}}^{-1}\left(\frac{\lambda}{2}\overline{\mathbf{X}}-\widetilde{\mathbf{s}}\right) \tag{5}$$

By substituting the value of $\widetilde{\mathbf{w}}$, we get:

$$\mathbb{E}\left[R_P\right]-\mathbb{E}\left[R_1\right]=\widetilde{\mathbf{w}}'\overline{\mathbf{X}}\Longleftrightarrow\mathbb{E}\left[R_P\right]-\mathbb{E}\left[R_1\right]=\frac{\lambda}{2}\overline{\mathbf{X}}'.\widetilde{\mathbf{V}}^{-1}.\overline{\mathbf{X}}-\overline{\mathbf{X}}'.\widetilde{\mathbf{V}}^{-1}.\widetilde{\mathbf{s}} \tag{6}$$

Denote : $\widetilde{A}=\overline{\mathbf{X}}'.\widetilde{\mathbf{V}}^{-1}.\widetilde{\mathbf{s}}$ and $\widetilde{B}=\overline{\mathbf{X}}'.\widetilde{\mathbf{V}}^{-1}.\overline{\mathbf{X}}$ (which are real numbers).

We get:

$$\frac{\lambda}{2}=\left(\frac{\mathbb{E}\left[R_P\right]-\mathbb{E}\left[R_1\right]+\widetilde{A}}{\widetilde{B}}\right) \tag{7}$$

By substituting the value of λ, we finally determine the vector of optimal weights.

Proposition 2 *(Optimal weights) The vector of weights of the minimum variance portfolios for each given level of return expectation* $\mathbb{E}\left[R_P\right]$ *is given by:*

$$\widetilde{\mathbf{w}}=\left(\frac{\mathbb{E}\left[R_P\right]-\mathbb{E}\left[R_1\right]+\widetilde{A}}{\widetilde{B}}\right)\widetilde{\mathbf{V}}^{-1}\overline{\mathbf{X}}-\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{s}} \tag{8}$$

We can rewrite (8) as follows:

$$\widetilde{\mathbf{w}}=\left(\frac{-\mathbb{E}\left[R_1\right]+\widetilde{A}}{\widetilde{B}}\right)\widetilde{\mathbf{V}}^{-1}\overline{\mathbf{X}}-\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{s}}+\mathbb{E}\left[R_P\right]\left(\frac{1}{\widetilde{B}}\right)\widetilde{\mathbf{V}}^{-1}\overline{\mathbf{X}} \tag{9}$$

Let us note respectively $\widehat{\mathbf{w}}_1$ and $\widehat{\mathbf{w}}_2$ the quantities:

$$\left(\frac{-\mathbb{E}\left[R_1\right]+\widetilde{A}}{\widetilde{B}}\right)\widetilde{\mathbf{V}}^{-1}\overline{\mathbf{X}}-\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{s}} \text{ and } \left(\frac{1}{\widetilde{B}}\right)\widetilde{\mathbf{V}}^{-1}\overline{\mathbf{X}}$$

We deduce the following result.

Proposition 3 *(Two-fund separation) The set of optimal solutions* $\widetilde{\mathbf{w}}$ *is a half-line defined by:*

$$\widetilde{\mathbf{w}}=\widehat{\mathbf{w}}_1+\mathbb{E}\left[R_P\right].\widehat{\mathbf{w}}_2 \tag{10}$$

Expression (8) of the optimal portfolio $\widetilde{\mathbf{w}}$ allows us to determine the relationship between the risk and the expected return of this portfolio.

Indeed, we have:

$$\begin{aligned}
\sigma^2\left(R_P\right) &= \sigma_{11}+2\widetilde{\mathbf{w}}'.\widetilde{\mathbf{s}}+\widetilde{\mathbf{w}}'\widetilde{\mathbf{V}}\widetilde{\mathbf{w}} \\
&= \sigma_{11}+2\widetilde{\mathbf{s}}'.\left[\frac{\lambda}{2}\widetilde{\mathbf{V}}^{-1}\overline{\mathbf{X}}+\widetilde{\mathbf{V}}^{-\mathbf{1}}\widetilde{\mathbf{s}}\right]+\left[\frac{\lambda}{2}\widetilde{\mathbf{V}}^{-1}\overline{\mathbf{X}}+\widetilde{\mathbf{V}}^{-\mathbf{1}}\widetilde{\mathbf{s}}\right].\widetilde{\mathbf{V}}.\left[\frac{\lambda}{2}\widetilde{\mathbf{V}}^{-1}\overline{\mathbf{X}}+\widetilde{\mathbf{V}}^{-\mathbf{1}}\widetilde{\mathbf{s}}\right] \\
&= \sigma_{11}-\widetilde{\mathbf{s}}'.\widetilde{\mathbf{V}}^{-\mathbf{1}}\widetilde{\mathbf{s}}+\left(\frac{\lambda}{2}\right)^2\left[\overline{\mathbf{X}}'\widetilde{\mathbf{V}}^{-1}\overline{\mathbf{X}}\right].
\end{aligned}$$

Denote:

$$\Gamma=\sigma_{11}-\widetilde{\mathbf{s}}'.\widetilde{\mathbf{V}}^{-\mathbf{1}}\widetilde{\mathbf{s}}$$

We get a characterization of the portfolio frontier.

Proposition 4 *(Portfolio frontier with factor investing) The relationship between the target expected return and the corresponding minimal variance is characterized by the following equation:*

$$\frac{\sigma^2\left(R_P\right)}{\Gamma}-\frac{\left(\mathbb{E}\left(R_P\right)-\left[\mathbb{E}\left[R_1\right]-\widetilde{A}\right]\right)^2}{\Gamma\widetilde{B}}=1 \tag{11}$$

This equation is that of a hyperbola with vertex $\left(\sqrt{\Gamma},\left[\mathbb{E}\left[R_1\right]-\widetilde{A}\right]\right)$.

2.2 Case with risky assets and a risk-free asset

When a risk-free asset is present on the market, a similar line of reasoning to the previous paragraph can be applied. The vector $\widetilde{\mathbf{w}}$ still represents the weights of the m risky factors, whose vector of profitability expectations is denoted $\overline{R}$. The risk-free asset yields the certain rate, denoted R_f. The proportion of wealth invested in the risk-free asset is denoted w_0. Thus, the budget constraint is written as follows:

$$w_0=1-w_1. \tag{12}$$

Denote $w=\binom{w_1}{\widetilde{\mathbf{w}}}$. Therefore, the optimization program corresponds to the Markowitz's problem in the presence of a risk-free asset (no constraint on the vector of weight $w=\binom{w_1}{\widetilde{\mathbf{w}}}$, the adjustment being made by taking $w_0=1-w_1$).

Note that the return expectation constraint here becomes:

$$w_0R_f+w_1\mathbb{E}\left[R_1\right]+\widetilde{\mathbf{w}}'\overline{\mathbf{X}}=\mathbb{E}\left[R_P\right]$$

which is equivalent to:

$$w_1 \left(\mathbb{E}\left[R_1\right] - R_f\right) + \widetilde{\mathbf{w}}'\overline{\mathbf{X}} = \mathbb{E}\left[R_P\right] - R_f$$

The Markowitz's results with the risk-free asset can therefore be used with appropriate modifications:

1) $\mathbb{E}\left[R_1\right]$ is replaced by $\left(\mathbb{E}\left[R_1\right] - R_f\right)$. Denote:

$$\overline{\mathbf{X}_{\text{mod}}} = \begin{pmatrix} \mathbb{E}\left[R_1\right] - R_f \\ \mathbb{E}\left[X_2\right] \\ . \\ . \\ \mathbb{E}\left[X_{m+1}\right] \end{pmatrix} \text{ and } \mathbf{V}_{\text{mod}} = \begin{pmatrix} \sigma_{11}^X & . & . \;\; . & \sigma_{1(m+1)}^X \\ . & \sigma_{22}^X & & \sigma_{2(m+1)}^X \\ . & & . & . \\ . & & . & . \\ \sigma_{(m+1)1}^X & \sigma_{(m+1)2}^X & . \;\; . & \sigma_{(m+1)(m+1)}^X \end{pmatrix}$$

2) The Lagrangian of the optimization problem is now written as:

$$L\left(\mathbf{w},\lambda\right) = \mathbf{w}'\mathbf{V}_{\text{mod}}\mathbf{w} + \lambda\left(\mathbb{E}\left[R_P\right] - R_f - \mathbf{w}'\overline{\mathbf{X}_{\text{mod}}}\right) \tag{13}$$

The constrained optimization problem (13) becomes the following free optimization problem:

$$\underset{\{\mathbf{w},\lambda\}}{Min}\; L\left(\mathbf{w},\lambda\right) = \mathbf{w}'\mathbf{V}_{\text{mod}}\mathbf{w} + \lambda\left(\mathbb{E}\left[R_P\right] - R_f - \mathbf{w}'\overline{\mathbf{X}_{\text{mod}}}\right) \tag{14}$$

The necessary and sufficient first-order conditions for a global minimum are:

$$\frac{\partial L\left(\mathbf{w},\lambda\right)}{\partial \mathbf{w}} = 2\mathbf{V}_{\text{mod}}\mathbf{w} - \lambda\overline{\mathbf{X}_{\text{mod}}} = 0 \tag{15}$$

$$\frac{\partial L\left(\mathbf{w},\lambda\right)}{\partial \lambda} = \mathbb{E}\left[R_P\right] - R_f - \mathbf{w}'\overline{\mathbf{X}_{\text{mod}}} = 0 \tag{16}$$

We get:

$$\begin{aligned} \mathbf{w} &= \frac{\lambda}{2}\mathbf{V}_{\text{mod}}^{-1}\left(\overline{\mathbf{X}_{\text{mod}}}\right) \\ &= \left(\frac{\mathbb{E}\left[R_P\right] - R_f}{\overline{\mathbf{X}_{\text{mod}}}'.\mathbf{V}_{\text{mod}}^{-1}.\overline{\mathbf{X}_{\text{mod}}}}\right)\mathbf{V}_{\text{mod}}^{-1}\left(\overline{\mathbf{X}_{\text{mod}}}\right) \end{aligned} \tag{17}$$

The variance of a portfolio return on the new frontier is written as:

$$\sigma^2\left(R_P\right) = \mathbf{w}'\mathbf{V}_{\text{mod}}\mathbf{w} = \left(\frac{\mathbb{E}\left[R_P\right] - R_f}{\overline{\mathbf{X}_{\text{mod}}}'.\mathbf{V}_{\text{mod}}^{-1}.\overline{\mathbf{X}_{\text{mod}}}}\right)^2 \left(\overline{\mathbf{X}_{\text{mod}}}\right)'\mathbf{V}_{\text{mod}}^{-1}\left(\overline{\mathbf{X}_{\text{mod}}}\right) = \frac{\left(\mathbb{E}\left[R_P\right] - R_f\right)^2}{J_{\text{mod}}} \tag{18}$$

with $J_{\text{mod}} = \overline{\mathbf{X}_{\text{mod}}}'.\mathbf{V}_{\text{mod}}^{-1}.\overline{\mathbf{X}_{\text{mod}}}$.

Proposition 5 *(Portfolio frontier with a risk-free asset and factor investing) The standard devia-*

tion of portfolio return on the new frontier is given by:

$$\sigma(R_P) = \begin{cases} \frac{\left(\mathbb{E}[R_P]-R_f\right)}{\sqrt{J_{\text{mod}}}} & \text{if } \mathbb{E}[R_P] \geq R_f \\ -\frac{\left(\mathbb{E}[R_P]-R_f\right)}{\sqrt{J_{\text{mod}}}} & \text{if } \mathbb{E}[R_P] < R_f \end{cases} \tag{19}$$

In the presence of a risk-free asset, the minimum variance portfolio frontier is the union of two half-lines whose origin is the point with coordinates $(0, R_f)$ *in the standard deviation expected return plane with slopes* $\sqrt{J_{\text{mod}}}$ *and* $-\sqrt{J_{\text{mod}}}$.

In what follows, we search the tangency portfolio in the factor investing framework. By substituting $\mathbf{w}$ from (17), we get : (note that here condition $w_0 = 0$ is equivalent to $w_1 = 1$)

$$(\mathbf{1}, \mathbf{0}, ..., \mathbf{0}).\mathbf{V}_{\text{mod}}^{-1}\left(\overline{\mathbf{X}_{\text{mod}}}\right)\frac{\mathbb{E}[R_P]-R_f}{J_{\text{mod}}} = 1 \tag{20}$$

$$\Longleftrightarrow \mathbb{E}[R_P]-R_f = \frac{J_{\text{mod}}}{(\mathbf{1}, \mathbf{0}, ..., \mathbf{0}).\mathbf{V}_{\text{mod}}^{-1}\left(\overline{\mathbf{X}_{\text{mod}}}\right)} \tag{21}$$

Finally, by substituting this expression of $\mathbb{E}[R_P]-R_f$ in (17), we deduce:

$$\mathbf{w}_t = \frac{J_{\text{mod}}}{(\mathbf{1}, \mathbf{0}, ..., \mathbf{0}).\mathbf{V}_{\text{mod}}^{-1}\left(\overline{\mathbf{X}_{\text{mod}}}\right)}\frac{\mathbf{V}_{\text{mod}}^{-1}\left(\overline{\mathbf{X}_{\text{mod}}}\right)}{\overline{\mathbf{X}_{\text{mod}}}'.\mathbf{V}_{\text{mod}}^{-1}.\overline{\mathbf{X}_{\text{mod}}}} \tag{22}$$

Since we have $J_{\text{mod}} = \overline{\mathbf{X}_{\text{mod}}}'.\mathbf{V}_{\text{mod}}^{-1}.\overline{\mathbf{X}_{\text{mod}}}$, we deduce the following result (see alternative proof in the Appendix).

Proposition 6 *(Tangency portfolio with factor investing)*

$$\mathbf{w}_t = \frac{\mathbf{V}_{\text{mod}}^{-1}\left(\overline{\mathbf{X}_{\text{mod}}}\right)}{(\mathbf{1}, \mathbf{0}, ..., \mathbf{0}).\mathbf{V}_{\text{mod}}^{-1}\left(\overline{\mathbf{X}_{\text{mod}}}\right)} \tag{23}$$

In what follows, we numerically illustrate some properties of the optimal frontier. In particular, we show why the combination of factor portfolios originally built by Fama and French (1992) induces mean-variance performances similar to that of optimized long-short portfolios, as emphasized by Brière and Szafarz (2017a,b, 2021) and illustrated in next section (see Figure 5). To this end, we consider the following first numerical base case with five risky assets R_i. We set the expectations and the standard deviations of the returns:

$$\begin{aligned} R_f &= 0.02; \overline{R_1} = 0.10; \overline{R_2} = 0.11; \overline{R_3} = 0.09; \overline{R_4} = 0.08; \overline{R_5} = 0.07; \\ \sigma_{R_1} &= 0.18; \sigma_{R_2} = 0.20; \sigma_{R_3} = 0.17; \sigma_{R_4} = 0.12; \sigma_{R_5} = 0.10. \end{aligned}$$

The correlations are given by:

$$\begin{aligned} \rho_{1,2} &= 0.7; \rho_{1,3} = 0.9; \rho_{1,4} = 0.6; \rho_{1,5} = 0.5; \rho_{2,3} = 0.75; \\ \rho_{2,4} &= 0.6; \rho_{2,5} = 0.55; \rho_{3,4} = 0.65; \rho_{3,5} = 0.5; \rho_{4,5} = 0.8. \end{aligned}$$

Figures 1 and 2 illustrate this numerical base case.

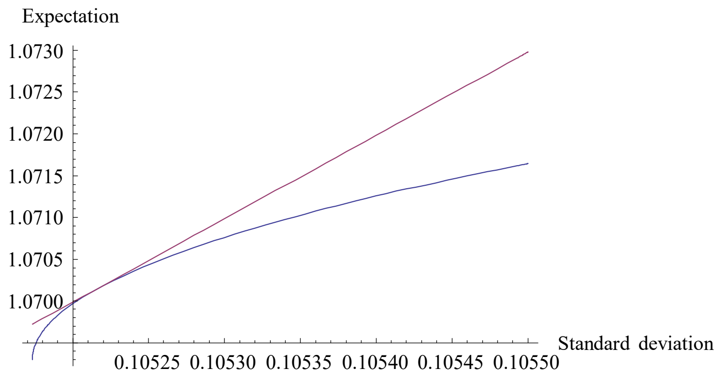


Figure 1: Tangency of the two portfolio frontiers with and without the risk-free asset in the factor investing framework

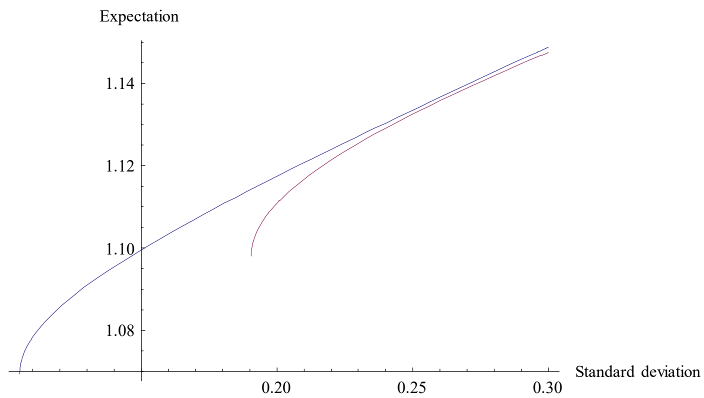


Figure 2: Comparison with the optimal frontier corresponding to no restriction on the weights of the five assets (without the risk-free asset)

Figure 1 shows the tangency of the two portfolio frontiers in the factor investing framework with $w_1 = 1$ and long-short positions. They correspond respectively to the cases of availability or non-availability of the risk-free asset. The two curves are very similar to the standard mean-variance analysis (respectively a hyperbola and a half-line). Figure 2 displays the portfolio frontiers without the risk-free asset corresponding to no restriction on R_i return weights and to factor investing with $w_1 = 1$ and long-short positions. For this numerical example, these two curves are fairly close for standard deviations within $[0.25; 0.30]$. In what follows, we study the proximity of the two portfolio frontiers corresponding to the case of long-short strategies and the case with no restriction on the weights of the five assets.

We focus on the impact of correlations. To this end, we assume that all correlations of risky returns R_i are equal and set four different values, namely $\rho = 0$, $\rho = 0.3$, $\rho = 0.6$, $\rho = 0.9$. Figure 3 illustrates this numerical base case.

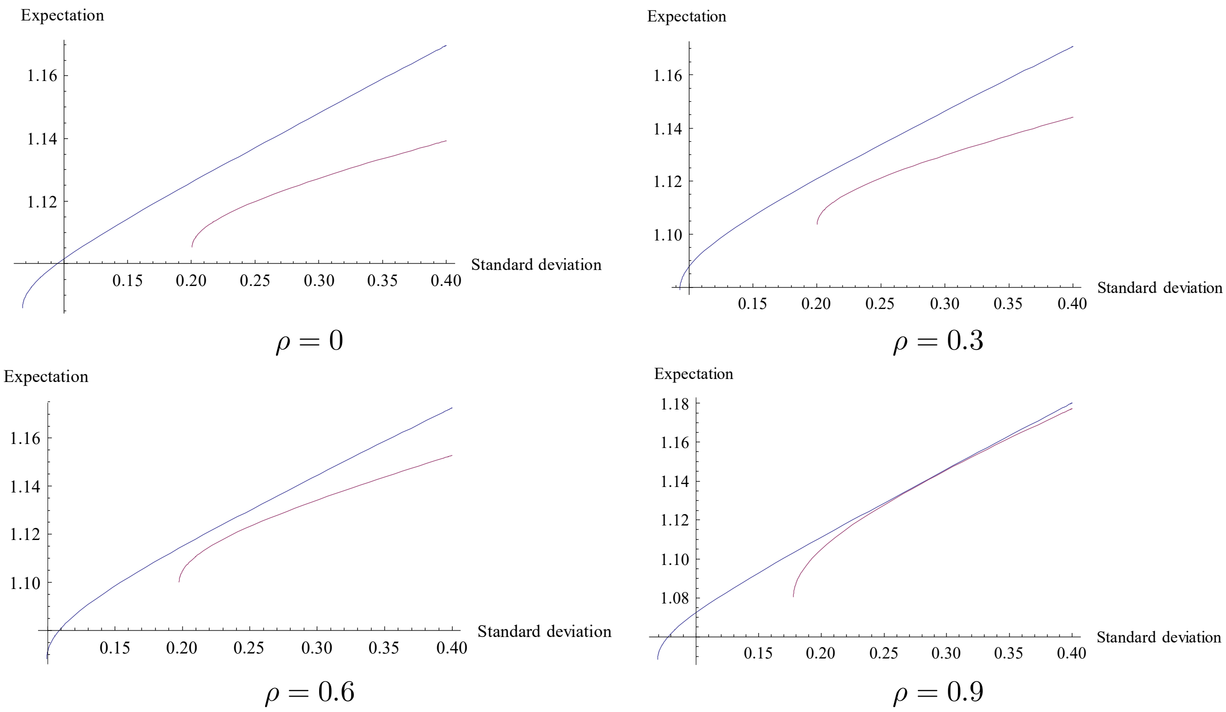


Figure 3: Closeness of the two portfolio frontiers corresponding to the long-short strategies case and the case with no restriction on the weights of the five assets - Impact of the common correlation

Figure 3 shows that, for a correlation $\rho = 0.9$, the two portfolio frontiers are very close (even though they do not intersect). Looking at Table 2 in the next section, we can see that it corresponds to usual values when dealing with the ten factors that are defined as the long and short legs of the five factors in the Fama-French (2015) model. Note also that the correlations of the five factors of the Fama-French (2015) have low absolute values meaning that they are almost orthogonal (see Table 3 in next section). This can be explained by the following proposition.

Proposition 7 *(Orthogonality property) Consider four random variables X, Y, Z and T with respective standard deviations σ_X, σ_Y, σ_Z, σ_T and correlations $\rho_{X,Z}$, $\rho_{X,T}$, $\rho_{Y,Z}$ and $\rho_{Y,T}$. Then, we get*

$$Cov(X - Y, Z - T) = \rho_{X,Z}\sigma_X\sigma_Z - \rho_{X,T}\sigma_X\sigma_T - \rho_{Y,Z}\sigma_Y\sigma_Z + \rho_{Y,T}\sigma_Y\sigma_T.$$

Thus, if all the standard deviations are equal (i.e. $\sigma_X = \sigma_Y = \sigma_Z = \sigma_T$) and all the correlation coefficients are equal (i.e. $\rho_{X,Z} = \rho_{X,T} = \rho_{Y,Z} = \rho_{Y,T}$) then $Cov(X - Y, Z - T) = 0$.

2.3 Minimization of the distance between the two efficient frontiers

2.3.1 Standard mean-variance analysis

Recall the standard mean-variance analysis for the asset returns $\mathbf{R}$ with

$$\overline{\mathbf{R}} = \begin{pmatrix} \mathbb{E}[R_1] \\ . \\ . \\ . \\ \mathbb{E}[R_n] \end{pmatrix} \text{ and } \mathbf{V} = \begin{pmatrix} \sigma_{11}^R & . & . & . & \sigma_{1n}^R \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ \sigma_{n1}^R & . & . & . & \sigma_{nn}^R \end{pmatrix}$$

The efficient frontier is characterized by the following hyperbola's equation:

$$\frac{\sigma^2(\alpha)}{\frac{1}{C}} - \frac{(\alpha - \frac{A}{C})^2}{\frac{D}{C^2}} = 1 \tag{24}$$

with

$$A = \bar{R}' \Sigma^{-1} 1, B = \bar{R}' \Sigma^{-1} \overline{R}, C = 1' \Sigma^{-1} 1, D = BC - A^2$$

2.3.2 Spread between two hyperbolas

Consider the two following hyperbolas $\mathcal{H}_1$ and $\mathcal{H}_2$ defined from equations:

$$\mathcal{H}_1 : \frac{x^2}{a^2} - \frac{(y-c)^2}{b^2} = 1 \text{ and } \mathcal{H}_2 : \frac{x^2}{\alpha^2} - \frac{(y-\gamma)^2}{\beta^2} = 1, \tag{25}$$

with $a > 0$, $b > 0$, $c \geq 0$, $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$.

In this first quadrant (i.e. $x \geq 0$ and $y \geq 0$), previous equations are equivalent to:

$$\mathcal{H}_1 : y = c + b\sqrt{\frac{x^2}{a^2} - 1} \text{ and } \mathcal{H}_2 : y = \gamma + \beta\sqrt{\frac{x^2}{\alpha^2} - 1}. \tag{26}$$

We assume that $\mathcal{H}_1$ is always above $\mathcal{H}_2$ for $x > 0$ and $y > 0$. In particular, this implies that $\alpha > a$ (comparison of the x-axis of the summits of hyperbolas) and $c + b\sqrt{\frac{\alpha^2}{a^2} - 1} > \gamma$ (comparison of the y-axis of the summit of hyperbola $\mathcal{H}_2$ with the corresponding point on hyperbola $\mathcal{H}_1$). Comparing the asymptotes, we must also have: $\frac{b}{a} > \frac{\beta}{\alpha}$.

We search to minimize the spread for $x > a$:

$$c + b\sqrt{\frac{x^2}{a^2} - 1} - \left(\gamma + \beta\sqrt{\frac{x^2}{\alpha^2} - 1}\right) = (c - \gamma) + b\sqrt{\frac{x^2}{a^2} - 1} - \beta\sqrt{\frac{x^2}{\alpha^2} - 1}.$$

Examine now the function $\varphi(x) = (c - \gamma) + b\sqrt{\frac{x^2}{a^2} - 1} - \beta\sqrt{\frac{x^2}{\alpha^2} - 1}$.

We have:

$$\varphi'(x) = \frac{b}{a^2}\frac{x}{\sqrt{\frac{x^2}{a^2} - 1}} - \frac{\beta}{\alpha^2}\frac{x}{\sqrt{\frac{x^2}{\alpha^2} - 1}}.$$

To find the minimum of φ, we need to solve the following equation $\varphi'(x) = 0$, which is equivalent to (here $x > 0$):

$$\frac{b}{a^2}\frac{1}{\sqrt{\frac{x^2}{a^2}-1}} = \frac{\beta}{\alpha^2}\frac{1}{\sqrt{\frac{x^2}{\alpha^2}-1}},$$

which yields to:

$$\left(\frac{x^2}{\alpha^2}-1\right)\left(\frac{b\alpha^2}{\beta a^2}\right)^2 = \left(\frac{x^2}{a^2}-1\right).$$

Denote $d = \left(\frac{b\alpha^2}{\beta a^2}\right)^2$. Since we have $\alpha > a$ and $\frac{b}{a} > \frac{\beta}{\alpha}$, we deduce that $d > \frac{\alpha^2}{a^2} > 1$. The previous equation is equivalent to:

$$\left(\frac{d}{\alpha^2}-\frac{1}{a^2}\right)x^2 = d-1.$$

Finally, we obtain the solution to the equation $\varphi'(x) = 0$. It is given by:

$$x^* = \frac{\alpha\sqrt{(d-1)}}{\sqrt{\left(d-\frac{\alpha^2}{a^2}\right)}} = \frac{a\sqrt{\left(\frac{b^2}{\beta^2}\left(\frac{\alpha^2}{a^2}\right)^2-1\right)}}{\sqrt{\left(\frac{b^2\alpha^2}{\beta^2 a^2}-1\right)}}. \tag{27}$$

We deduce that the minimal spread is given by:

$$\varphi(x^*) = (c-\gamma) + b\sqrt{\frac{\left(\frac{b^2}{\beta^2}\left(\frac{\alpha^2}{a^2}\right)^2-1\right)}{\left(\frac{b^2\alpha^2}{\beta^2 a^2}-1\right)}-1} - \beta\sqrt{\frac{a^2}{\alpha^2}\frac{\left(\frac{b^2}{\beta^2}\left(\frac{\alpha^2}{a^2}\right)^2-1\right)}{\left(\frac{b^2\alpha^2}{\beta^2 a^2}-1\right)}-1}. \tag{28}$$

In the framework of the mean-variance analysis with specific long-short strategies, we set:

$$a = \frac{1}{\sqrt{C}},\ b = \frac{\sqrt{D}}{C},\ c = \frac{A}{C} \text{ and } \alpha = \sqrt{\Gamma},\ \beta = \sqrt{\Gamma\widetilde{B}},\ \gamma = \left[\mathbb{E}[R_1] - \widetilde{A}\right],$$

where the hyperbolas $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined respectively by Equations 24 and 11.

We deduce that the spread is minimal at:

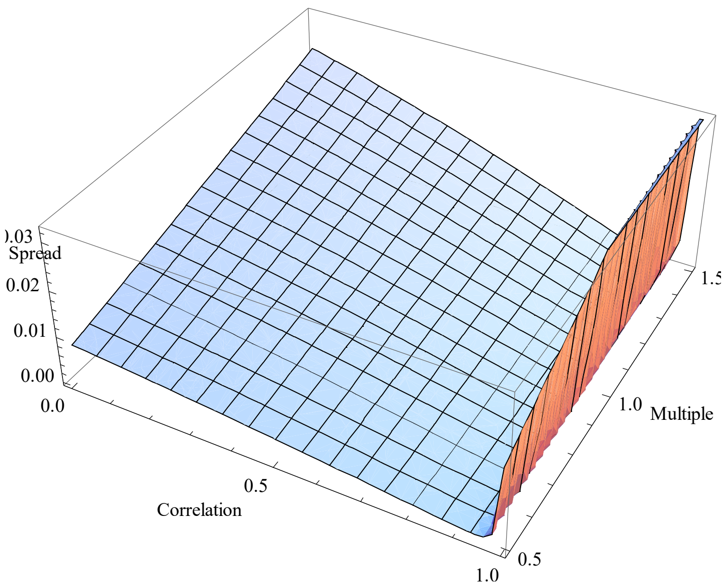
$$\sigma^* = \frac{\sqrt{\left(\frac{D\Gamma}{\widetilde{B}}-1\right)}}{\sqrt{\left(\frac{D}{\widetilde{B}}-C\right)}}. \tag{29}$$

The minimal spread is given by:

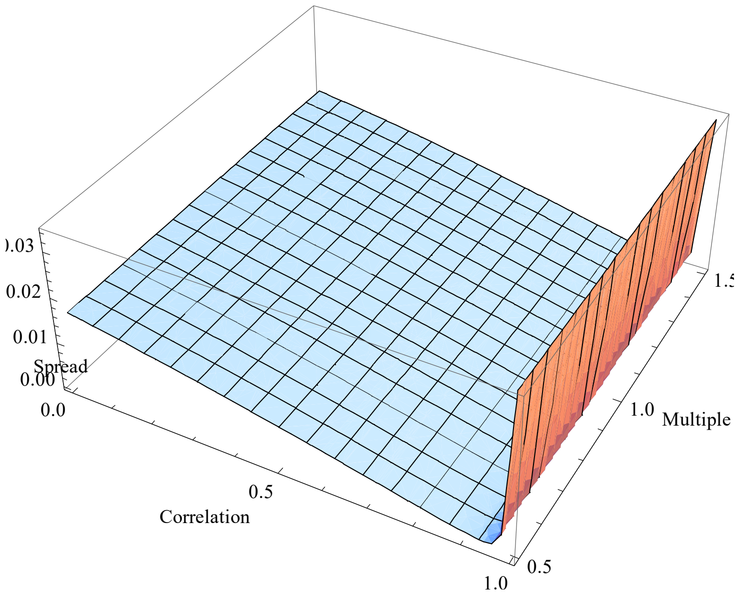
$$\varphi(\sigma^*) = \left(\frac{A}{C} - \left[\mathbb{E}[R_1]-\widetilde{A}\right]\right) + \frac{\sqrt{D}}{C}\sqrt{C\frac{\left(\frac{D\Gamma}{\widetilde{B}}-1\right)}{\left(\frac{D}{\widetilde{B}}-C\right)}-1} - \sqrt{\Gamma\widetilde{B}}\sqrt{\frac{1}{\Gamma}\frac{\left(\frac{D\Gamma}{\widetilde{B}}-1\right)}{\left(\frac{D}{\widetilde{B}}-C\right)}-1}. \tag{30}$$

Using our numerical base case, we can illustrate the behavior of the spread as a function of the level of the correlation and various levels of drift and volatility (see Figure 4). We can also analyze the spread as a function of correlation for the market and two pairs of factors (see Figure

6 in next section). In what follows, we analyze the minimum spread between the two optimal frontiers for our specific numerical example (see details on the explicit value of the spread between two hyperbolas in the appendix). We start by multiplying the drift by the same multiple between −50% and +150%. Next, we multiply volatility by the same multiple between −50% and +150%. Note that the optimal frontiers we have obtained are not simply homothetic (due for example to the C term in the hyperbola equation 24). Figure 4 shows that the minimum is reached most often around 0.9.



Spread between the two frontiers as a function of the correlation and the drifts



Spread between the two frontiers as a function of the correlation and the volatilities

Figure 4: Minimal spread of the two portfolio frontiers as a function of the correlation

3 Optimal portfolio allocation with Fama-French factors

In what follows, we proceed with an empirical illustration of factor investing on long and short legs. Since unlimited short sales are not realistic, Brière and Szafarz (2017a,b, 2021) consider several optimization programs which they solved numerically[8]:

1. Global Long-Only Efficient Frontier in which any short position is excluded;
2. Long-Short Market + Long-Only factors Efficient Frontier for which there is no restriction on market exposure but short positions in factors are not allowed;
3. 130/30 portfolios Efficient Frontier which implies a 130% long position and a 30% short position;
4. Efficient Frontier without constraint on the weights.
5. Fama-French long-short factors (i.e. FF Benchmark);

In this paper, we focus on optimization programs 4 and 5. Recall that, in the previous section, we went a step further by proving the existence of an analytical solution to program 5.

For the investment universe and in addition to the market, Brière and Szafarz (2017a,b, 2021) "assets" are defined as the long and short legs of the risk factors introduced by Fama-French (1992, 2015) and Carhart (1997): (1) small, (2) big, (3) value, (4) growth, (5) robust profitability, (6) weak profitability, (7) conservative investment, (8) aggressive investment, (9) high momentum, (10) low momentum. From the data retrieved from Kenneth French's website, we can recover these ten factors. We use monthly gross total return (in USD) for all variables over the period July 1963-December 2019.

In what follows, we use the following notations:
$R_{i,t}$ is the return of portfolio i;
$R_{f,t}$ is the risk-free rate;
$R_{M,t} - R_{f,t}$ is the return spread between the capitalization-weighted stock market and cash;
SMB_t is the return spread of small minus large stocks (i.e. the size effect);
HML_t is the return spread of cheap minus expensive stocks (i.e. the value effect);
RMW_t is the return spread of the most profitable firms minus the least profitable;
CMA_t is the return spread of firms that invest conservatively minus aggressively;
MOM_t is the return spread of low momentum minus high momentum stocks.

[8]Program 4 is the Markowitz's case and has an analytical solution.

Table 1 displays the descriptive statistics for the T-bills, the market and the ten factors on the entire period. We note that the means of the factors are approximately between 8% and 15% while the volatilities are approximately between 15% and 22%.

Table 1: Statistics for Fama-French Factors Anualized Returns in percentage (1963:07-2019:12)

	Rf	Mkt	Small	Big	Value	Growth	Robust Profit	Weak Profit	Cons. Invt.	Aggr. Invt	High Mom.	Low Mom.
Mean	4.54	11.04	14.02	11.28	14.65	10.99	13.69	10.58	14.16	10.88	16.15	8.41
Volatility	0.91	15.14	19.91	14.89	17.24	18.65	16.82	19.17	16.86	19.23	18.11	21.48
Skewness	0.65	-0.52	-0.46	-0.44	-0.47	-0.48	-0.56	-0.52	-0.52	-0.52	-0.63	0.34
Kurtosis	3.73	4.97	5.42	4.89	6.16	4.73	5.36	4.97	5.21	4.78	5.34	6.89
Min	0.00	-22.64	-29.54	-21.45	-23.54	-27.77	-25.78	-27.79	-25.44	-27.83	-27.87	-24.78
Max	1.35	16.61	27.12	16.66	25.95	17.92	20.38	21.27	20.21	21.08	17.49	39.93

Notes: The minimum and maximum are monthly returns, not annualized.

Table 2 shows the correlation matrix of all variables. We note that the correlations between factors are approximately between 0.85 and 0.99 and most often close to 0.9.

Table 2: Correlation Matrix Long and Short Leg Fama-French Factors (1963:07 -2019:12)

	Rf	Mkt	Small	Big	Value	Growth	Robust Profit	Weak Profit	Cons. Invt.	Aggr. Invt	High Mom.	Low Mom.
Rf	1.000	-0.024	-0.030	-0.014	0.003	-0.042	-0.029	-0.028	-0.017	-0.041	-0.008	-0.050
Mkt	-0.024	1.000	0.887	0.993	0.891	0.952	0.961	0.933	0.935	0.957	0.919	0.868
Small	-0.030	0.887	1.000	0.860	0.931	0.946	0.946	0.961	0.964	0.953	0.928	0.879
Big	-0.014	0.993	0.860	1.000	0.899	0.921	0.947	0.911	0.926	0.930	0.891	0.864
Value	0.003	0.891	0.931	0.899	1.000	0.856	0.916	0.912	0.954	0.882	0.863	0.878
Growth	-0.042	0.952	0.946	0.921	0.856	1.000	0.963	0.957	0.933	0.990	0.944	0.866
R.P.	-0.029	0.961	0.946	0.947	0.916	0.963	1.000	0.922	0.947	0.970	0.936	0.880
W.P.	-0.028	0.933	0.961	0.911	0.912	0.957	0.922	1.000	0.963	0.963	0.926	0.889
C.I.	-0.017	0.935	0.964	0.926	0.954	0.933	0.947	0.963	1.000	0.935	0.923	0.885
A.I.	-0.041	0.957	0.953	0.930	0.882	0.990	0.970	0.963	0.935	1.000	0.945	0.883
H.M.	-0.008	0.919	0.928	0.891	0.863	0.944	0.936	0.926	0.923	0.945	1.000	0.744
L.M.	-0.050	0.868	0.879	0.864	0.878	0.866	0.880	0.889	0.885	0.883	0.744	1.000
Aver. Corr.	**-0.025**	**0.843**	**0.839**	**0.830**	**0.817**	**0.844**	**0.851**	**0.846**	**0.850**	**0.851**	**0.819**	**0.781**

Table 3 shows the correlation matrix of the Market and the 5 factors. As expected, they all correlate weakly and even mostly negatively, with the exception of the correlation of HML with CMA, which is equal to 0.69. This characteristic fully justifies their use in a diversified portfolio.

Table 3: Correlation Matrix Fama-French Factors (1963:07 -2019:12)

	Mkt	SMB	HML	RMW	CMA	Mom
	Mkt	SMB	HML	RMW	CMA	Mom
Mkt	1.000	0.276	-0.247	-0.228	-0.382	-0.138
SMB	0.276	1.000	-0.064	-0.347	-0.104	-0.035
HML	-0.247	-0.064	1.000	0.063	0.695	-0.196
RMW	-0.228	-0.347	0.063	1.000	-0.034	0.110
CMA	-0.382	-0.104	0.695	-0.034	1.000	-0.027
MOM	-0.138	-0.035	-0.196	0.110	-0.027	1.000

In Figure 5, we plot the various efficient frontiers estimated on the entire sample and obtained by numerical optimization. We also indicate the points corresponding to the market and the ten factors. It is remarkable that the efficient frontier obtained from the market and the five long-short factors is so close to that obtained from the market and the ten short and long legs of each of these five factors. This is consistent with the numerical results in the previous section (see e.g. Figure 3), since their correlations are close to 0.9 (see Table 2). We also plot the two maximum Sharpe Ratio portfolios for optimization programs 4 and 5, along with the corresponding Capital Market Lines. The slope of CML 1 is equal to 1.376 which is only slightly higher than the slope of CML 2 which is equal to 1.276. This shows once again that using the Fama and French factors instead of the short and long legs of the same factors does not significantly worsen an investor's risk-return trade-off.

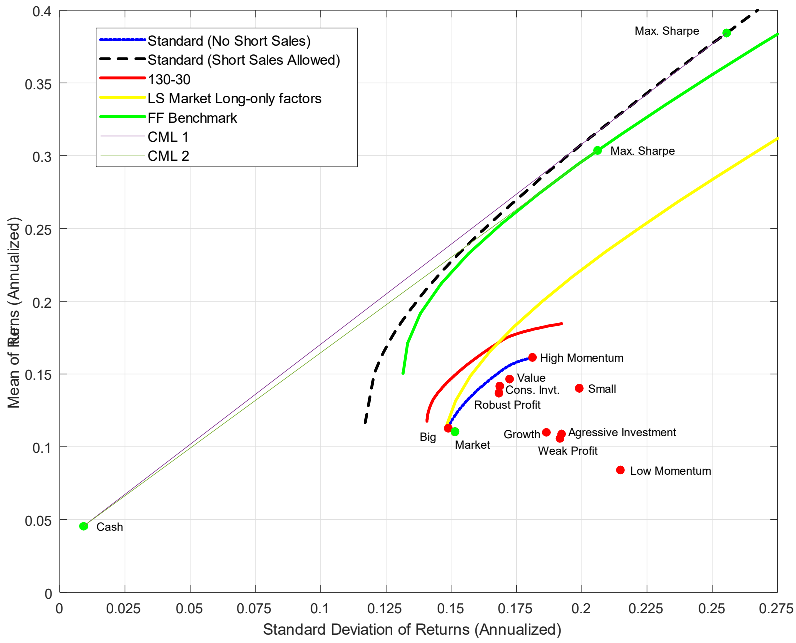


Figure 5: Efficient Frontiers and CMLs: 01/07/1963 to 01/12/2019

Tables 4 and 5 show that recessions do not alter the level of correlations between the long and short legs of the factors. The average correlation between the ten legs during a recession is 0.942 and 0.911 during a normal period (no recession). Over the whole period, the average correlation between the ten legs is equal to 0.917. As expected, the correlation increases slightly during recessions but the effect is not very significant. Moreover, the level of correlations during non-recessions is sufficient to justify the proximity of the efficient frontiers.

To assess the robustness of the proximity of optimal frontiers with respect to correlations, we examine their spread as a function of correlation for the market and two pairs of factors (using the explicit results on the spread between optimal frontiers provided in appendix). In particular, we choose the two "historical" pairs, namely the Small-Big and Value-Growth. Figure 6 shows the spreads. It shows that the minimum is still reached around 0.9. In all cases, the minimal spread is less than 0.5%.

Table 4: Correlation Matrix Long and Short Leg Fama-French Factors during Not Recession Periods (1963:07 -2019:12)

	Rf	Mkt	Small	Big	Value	Growth	Robust Profit	Weak Profit	Cons. Invt.	Aggr. Invt	High Mom.	Low Mom.
Rf	1.000	-0.004	-0.007	0.003	0.012	-0.014	0.003	-0.012	-0.004	-0.010	0.009	-0.025
Mkt	-0.004	1.000	0.874	0.992	0.885	0.947	0.957	0.923	0.927	0.952	0.911	0.862
Small	-0.007	0.874	1.000	0.841	0.924	0.941	0.936	0.957	0.961	0.947	0.931	0.868
Big	0.003	0.992	0.841	1.000	0.893	0.910	0.942	0.895	0.915	0.918	0.877	0.858
Value	0.012	0.885	0.924	0.893	1.000	0.847	0.914	0.902	0.952	0.870	0.860	0.871
Growth	-0.014	0.947	0.941	0.910	0.847	1.000	0.954	0.955	0.926	0.989	0.946	0.859
R.P.	0.003	0.957	0.936	0.942	0.914	0.954	1.000	0.909	0.939	0.962	0.934	0.871
W.P.	-0.012	0.923	0.957	0.895	0.902	0.955	0.909	1.000	0.958	0.959	0.922	0.882
C.I.	-0.004	0.927	0.961	0.915	0.952	0.926	0.939	0.958	1.000	0.927	0.916	0.882
A.I.	-0.010	0.952	0.947	0.918	0.870	0.989	0.962	0.959	0.927	1.000	0.950	0.869
H.M.	0.009	0.911	0.931	0.877	0.860	0.946	0.934	0.922	0.916	0.950	1.000	0.739
L.M.	-0.025	0.862	0.868	0.858	0.871	0.859	0.871	0.882	0.882	0.869	0.739	1.000
Aver. Corr.	**-0.004**	**0.839**	**0.834**	**0.822**	**0.812**	**0.842**	**0.848**	**0.841**	**0.845**	**0.848**	**0.818**	**0.776**

Notes: This table presents the correlations between the factors for the period identified by the NBER as not recession periods (see Federal Reserve Bank of St. Louis, 2023). According to the NBER, there are 636 months that do not correspond to economic recessions during the period July 1963: December 2019.

Table 5: Correlation Matrix Long and Short Leg Fama-French Factors during Recession Periods (1963:07 -2019:12)

	Rf	Mkt	Small	Big	Value	Growth	Robust Profit	Weak Profit	Cons. Invt.	Aggr. Invt	High Mom.	Low Mom.
Rf	1.000	-0.008	-0.048	0.013	0.035	-0.068	-0.063	-0.011	-0.006	-0.066	0.026	-0.091
Mkt	-0.008	1.000	0.921	0.997	0.906	0.968	0.971	0.960	0.962	0.970	0.951	0.894
Small	-0.048	0.921	1.000	0.909	0.951	0.957	0.972	0.971	0.974	0.969	0.927	0.923
Big	0.013	0.997	0.909	1.000	0.915	0.952	0.958	0.956	0.960	0.958	0.943	0.891
Value	0.035	0.906	0.951	0.915	1.000	0.880	0.920	0.940	0.960	0.909	0.889	0.902
Growth	-0.068	0.968	0.957	0.952	0.880	1.000	0.986	0.963	0.952	0.994	0.947	0.901
R.P.	-0.063	0.971	0.972	0.958	0.920	0.986	1.000	0.960	0.969	0.989	0.955	0.910
W.P.	-0.011	0.960	0.971	0.956	0.940	0.963	0.960	1.000	0.979	0.974	0.940	0.927
C.I.	-0.006	0.962	0.974	0.960	0.960	0.952	0.969	0.979	1.000	0.961	0.952	0.912
A.I.	-0.066	0.970	0.969	0.958	0.909	0.994	0.989	0.974	0.961	1.000	0.941	0.926
H.M.	0.026	0.951	0.927	0.943	0.889	0.947	0.955	0.940	0.952	0.941	1.000	0.800
L.M.	-0.091	0.894	0.923	0.891	0.902	0.901	0.910	0.927	0.912	0.926	0.800	1.000
Aver. Corr.	**-0.026**	**0.863**	**0.857**	**0.859**	**0.837**	**0.857**	**0.866**	**0.869**	**0.870**	**0.866**	**0.843**	**0.809**

Notes: This table presents the correlations between the factors for the period identified by the NBER as recession periods (see Federal Reserve Bank of St. Louis, 2023). According to the NBER, there are 85 months that correspond to economic recessions during the period July 1963: December 2019.

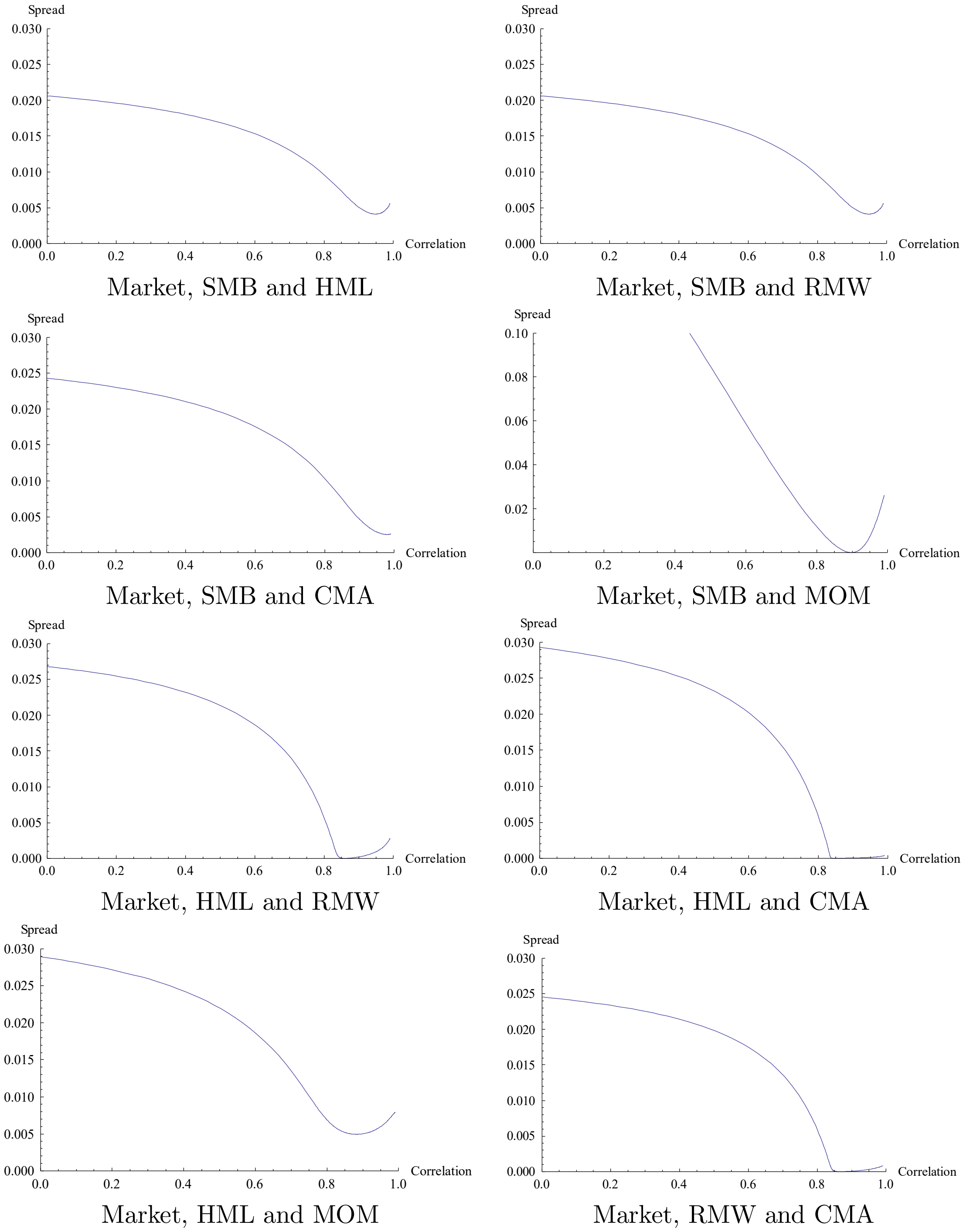


Figure 6: Spread as function of the correlation for the market and two pairs of factors

4 Conclusion

In this paper, we studied portfolio allocation optimality and the corresponding efficient frontiers in the factor-investing framework when using long-short strategies. Like Brière and Szafarz (2017a,b, 2021), using data retrieved from Kenneth French's website, we disentangled the long and short legs of the five historical factors considered by Fama and French (1992, 2015) and Carhart (1997) to obtain the corresponding ten factors. Next, we compared several portfolio frontiers corresponding to different constraints on portfolio weights. For a benchmark case such as that of the Fama-French frontier, we explicitly provided the optimal portfolio weights taking into account long-short positions. By examining the impact of correlations, we have justified why the efficient frontier generated by the market and the five long-short factors is very close to that obtained from the market and the ten short and long legs of each of these five factors. As a by-product, we have determined the explicit spreads between two Markowitz's frontiers when one frontier is dominated by the other. It should also be noted that the theoretical result on the use of long-short strategies obtained in this article can be applied to other financial issues, such as the analysis of "peer trading" strategies.

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Appendix

Alternative proof of the determination of the tangency portfolio with factor investing

By definition, the tangency portfolio belongs to the portfolio frontier constructed from the risky assets/factors alone and to the frontier with the risk-free asset defined from Equation (19). Thus, we have to search for the intersection:

$$\frac{\sigma^2(R_P)}{\Gamma} - \frac{\left(\mathbb{E}(R_P) - \left[\mathbb{E}[R_1] - \widetilde{A}\right]\right)^2}{\Gamma \widetilde{B}} = 1 \text{ with } \sigma(R_P) = \frac{(\mathbb{E}[R_P] - R_f)}{\sqrt{J_{\text{mod}}}}.$$

Then, we get:

$$\frac{(\mathbb{E}[R_P] - R_f)^2}{\Gamma J_{\text{mod}}} - \frac{\left(\mathbb{E}(R_P) - \left[\mathbb{E}[R_1] - \widetilde{A}\right]\right)^2}{\Gamma \widetilde{B}} = 1.$$

This a second order polynomial equation with respect to $x = \mathbb{E}[R_P]$.

We get its roots:

$$x^* = \frac{1}{\left(\frac{1}{\Gamma J_{\text{mod}}} - \frac{1}{\Gamma \widetilde{B}}\right)} \times$$

$$\left(\begin{array}{c} \left[R_f \frac{1}{\Gamma J_{\text{mod}}} - \left[\mathbb{E}[R_1] - \widetilde{A}\right] \frac{1}{\Gamma \widetilde{B}}\right] \\ \pm\sqrt{\left[R_f \frac{1}{\Gamma J_{\text{mod}}} - \left[\mathbb{E}[R_1] - \widetilde{A}\right] \frac{1}{\Gamma \widetilde{B}}\right]^2 - \left(\frac{1}{\Gamma J_{\text{mod}}} - \frac{1}{\Gamma \widetilde{B}}\right)\left(\frac{1}{\Gamma J_{\text{mod}}} R_f^2 - \frac{1}{\Gamma \widetilde{B}}\left[\mathbb{E}[R_1] - \widetilde{A}\right]^2 - 1\right)} \end{array} \right).$$

which is equivalent to:

$$x^* = \frac{\left[\widetilde{B} R_f - J_{\text{mod}}\left[\mathbb{E}[R_1] - \widetilde{A}\right]\right] \pm \sqrt{\Gamma J_{\text{mod}} \widetilde{B}} \sqrt{\left(\widetilde{B} - J_{\text{mod}}\right) + \left(\frac{1}{\Gamma}\right)\left(R_f - \left[\mathbb{E}[R_1] - \widetilde{A}\right]\right)^2}}{\left(\widetilde{B} - J_{\text{mod}}\right)}.$$

Note that we have:

$$\widetilde{B} - J_{\text{mod}} = \left(\overline{\mathbf{X}}'.\widetilde{\mathbf{V}}^{-1}.\overline{\mathbf{X}}\right) - \left(\overline{\mathbf{X}_{\text{mod}}}'.\mathbf{V}_{\text{mod}}^{-1}.\overline{\mathbf{X}_{\text{mod}}}\right) \text{ with } \overline{\mathbf{X}_{\text{mod}}} = \begin{pmatrix} \mathbb{E}[R_1] - R_f \\ \overline{\mathbf{X}} \end{pmatrix}$$

As a result, we get: (we retain only the largest root)

$$x^* = \frac{\left[\widetilde{B} R_f - J_{\text{mod}}\left[\mathbb{E}[R_1] - \widetilde{A}\right]\right] + \sqrt{\Gamma J_{\text{mod}} \widetilde{B}} \sqrt{\left(\widetilde{B} - J_{\text{mod}}\right) + \left(\frac{1}{\Gamma}\right)\left(R_f - \left[\mathbb{E}[R_1] - \widetilde{A}\right]\right)^2}}{\left(\widetilde{B} - J_{\text{mod}}\right)}.$$

Since we have:

$$\sqrt{\left(\widetilde{B} - J_{\text{mod}}\right) + \left(\frac{1}{\Gamma}\right)\left(R_f - \left[\mathbb{E}\left[R_1\right] - \widetilde{A}\right]\right)^2} = 0$$

we get:

$$x^* = \frac{J_{\text{mod}}\left[\mathbb{E}\left[R_1\right] - \widetilde{A}\right] - \widetilde{B}R_f}{\left(J_{\text{mod}} - \widetilde{B}\right)}.$$

Finally, the tangency portfolio can be deduced from this by applying:

$$\mathbf{w}^* = \left(\frac{x^* - R_f}{\overline{\mathbf{X}_{\text{mod}}}'.\mathbf{V}_{\text{mod}}^{-1}.\overline{\mathbf{X}_{\text{mod}}}}\right)\mathbf{V}_{\text{mod}}^{-1}\left(\overline{\mathbf{X}_{\text{mod}}}\right).$$